2017

Analytic Techniques for PDEs Fourier Series and the Method of Separation of Variables

Hend Elmhdi

Analytic Techniques for PDEs Fourier Series and the method of separation of variables

Hend Elmhdi

- **Thesis submitted to the Eberly College of Arts and Sciences at West Virginia University**

in partial fulfillment of the requirements for the degree of Master of Science in Mathematics

Charis Tsikkou PhD Chair

Harvey Diamond PhD

Marjorie Darrah PhD

Department of Mathematics

Morgantown, West Virginia 2017

Keywords: Fourier Series Technique

ABSTRACT

Analytic Techniques for PDEs Fourier Series and the method of separation of variables

Hend Elmhdi

The main topic of this study is deriving solutions of linear PDEs that have no mixed derivatives, using the method of separation of variables. One seeks a solution of a PDE, involving n independent variables, in a form of a product or sum and converts the PDE into n solvable ordinary differential equations to build the general solution out of these particular solutions.

Contans

Analytic Techniques for PDEs Fourier Series and the method of separation of variables

Hend Elmhdi

Abstract: The main topic of this study is the solution of linear PDEs that have no mixed derivatives using the method of separation of variables. One seeks a solution of a PDE involving n independent variables in a form of product or sum, and converts the PDE into n ordinary differential equations to build the general solution out of these particular solutions.

1-Fourier Series:

We will consider Fourier series and how important in engineering and other sciences. Function f can be represented by an infinite series of elementary trigonometric functions sines and cosines. We first need some background material. A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x and $f(x) = f(x + P) for\ all\ x$.

P is called the period of $f(x)$.

Example1: we have $\sin(x + 2\pi) = \sin x$ for all x, therefore $P = 2\pi$

Example2: $f(x) = \cot(x)$, $g(x) = \tan(x)$ have period π since

$\tan(x + \pi) = tax(x)$, $\cot(x + \pi) = \cot\,(x)$ for all x

We will consider the Fourier series and how the representation of some of the functions that are important in engineering sciences and other sciences Fourier series. We first need some background material. A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x. if we have a function $f(x)$, the function $f(x)$ called the function of a cycle, then we have $f(x \mp p)$

Ex 1: $f(x) = \sin(x)$, If $T = 2\pi$ then for every x we have $\sin(x + 2\pi) = \sin(x)$

Figure 1 sin x

Ex2: $f(x) = \cot(x)$, $f(x) = \tan(x)$ then $tan(x + \pi) = \tan(x)$, $cot(x + \pi) = \cot(x)$

Notes:

1- If $f(x)$ is a periodic function for T, then $f(x \pm nT) = f(x)$ for every $n \in N$

$\int_0^T f(x)dx = \int_c^{c+T} f(x)dx$ for every $c \in N$

Then $f(x)$ has period p it also has period $2p$

$f(x + 2p) = f([x + p] + p) = f(x + p) = f(x)$ then

$f(x + np) = f(x), for\ all\ x$

2- If $f(x)$ and $g(x)$have period p, then $af(x) + bg(x)$ with any constants $a\ and\ b$ also has the period p.

3- Simple function:

$cosx$	$sinx$
$cos2x$	$sin2x$
--	--
$cosnx$	$sinnx$

All these functions have period 2π, $\pi = 3.14$. The form is called ***trigonometric system***.

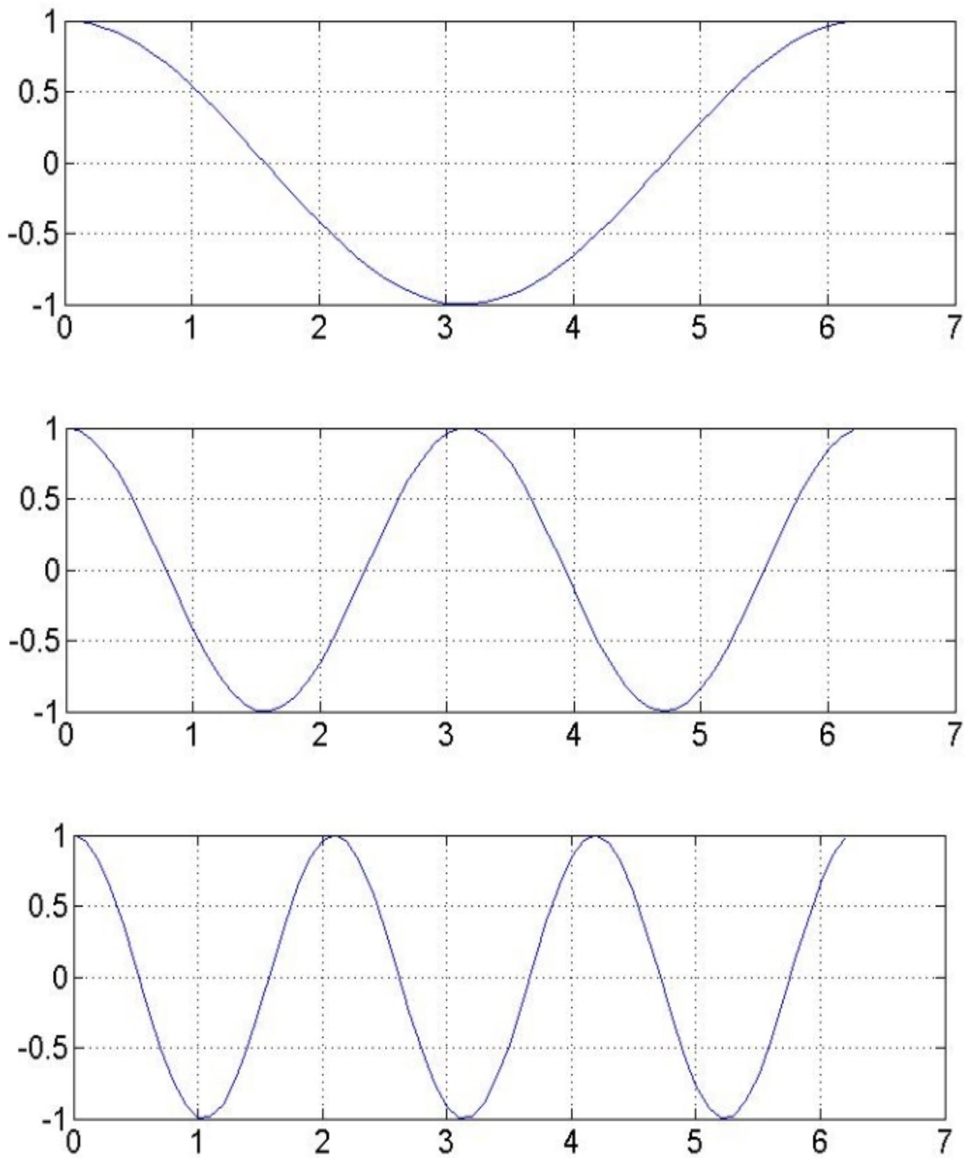

Figure 2 sin x, sins2x, sin3x

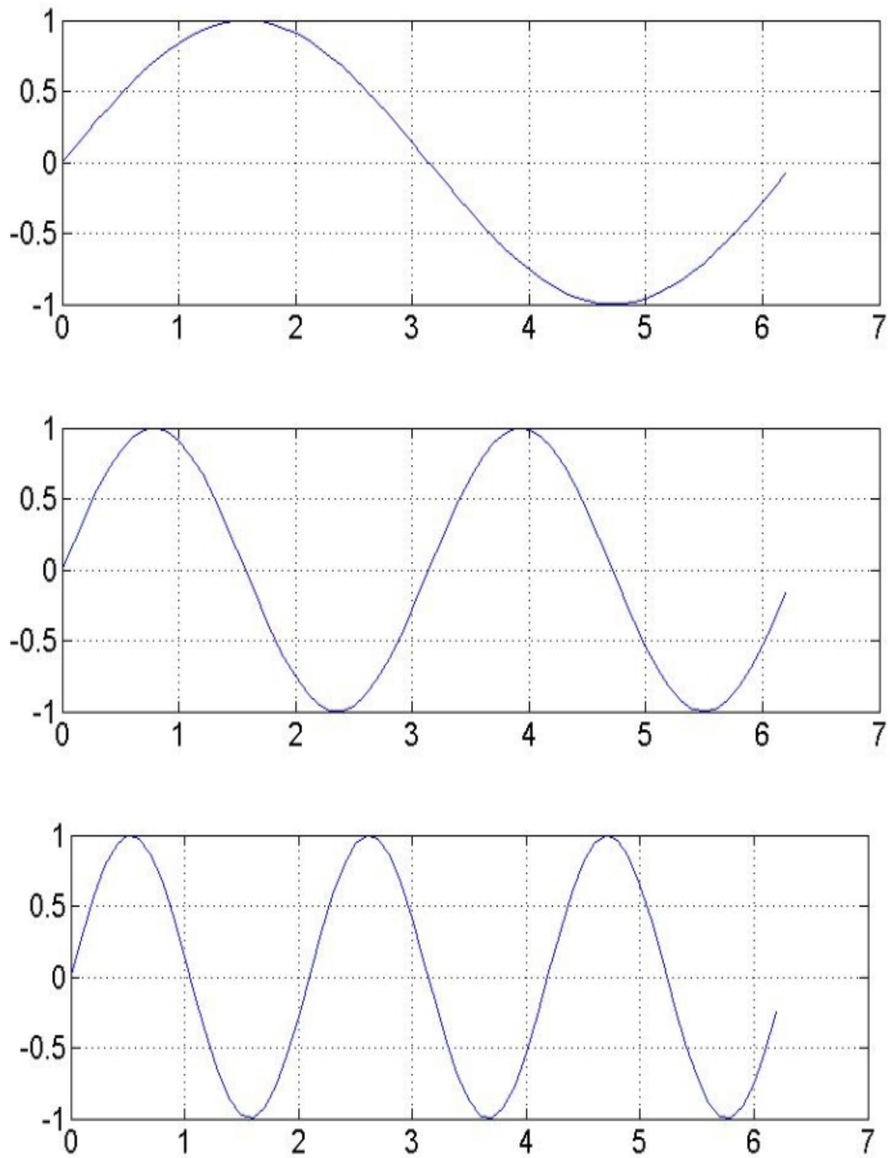

Figure 3 cosx, cos2x, cox3x

Cosine and Sine functions have the period 2π. Then we have now

$$f(x) = a_0 + a_1 cosx + b_1 sinx + a_2 cos2x + \ b_2 sin2x \pm - - - -$$

$$\boldsymbol{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)}$$

$a_0, a_1, b_1, a_2, b_2 ----$ are constants called coefficients of the series and each them has the period 2π. Suppose $f(x)$ dis a given function of period of 2π then using the equality sign

$\boldsymbol{f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}$ this is called the **Fourier Series** of $f(x)$ the coefficients of this series are called **Fourier Coefficients of $\boldsymbol{f(x)}$**

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \qquad (1)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\ dx \quad n = 1,2, \dots .. \qquad (2)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)\ dx \quad n = 1,2, \dots .. \qquad (3)$$

Basics Examples:

Ex. 1: Find the Fourier series expansion of the periodic function

$$f(x) = x, \quad -\pi \leq x \leq \pi, \ \ f(x+2\pi) = f(x)$$

Solution: $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0$ $\qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(nx)\ dx = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(nx)\ dx = \frac{2}{\pi}\int_{0}^{\pi} xsin(nx)\ dx = \frac{2}{n}(-1)^{n+1}$$

Therefore, the expansion of the given function on $[-\pi, \pi]$ is given by

$$x = 2[sinx - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots]$$

Ex. 2: Find the Fourier coefficients of the periodic function $f(x)$

$$f(x) = \begin{cases} -k & if -\pi < x < 0 \\ k & if\ 0 < x < \pi \end{cases} \qquad f(x+2\pi) = f(x)$$

Solution: $a_0 = 0$ the area under the curve of $f(x)$ between $-\pi\ and\ \pi$ is 0

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)\,dx = \frac{1}{\pi}[\int_{-\pi}^{0} -kcos(nx)dx +$

$\int_0^{\pi} kcos(nx)\,dx]$ $=\frac{1}{\pi}\left[-\frac{ksin(nx)}{n}|_{-\pi}^{0} + \frac{ksin(nx)}{n}|_0^{\pi}\right] = 0$ because $sinnx = 0\ at - \pi, 0$ and

all these cosine coefficients are 0. Then the Fourier series of $f(x)$ has no cosine terms,

just sine terms, it is a Fourier Sin series with coefficients $b_1, b_2, \dots\dots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)\,dx = \frac{1}{\pi}[\int_{-\pi}^{0} -ksin(nx)dx +$

$\int_0^{\pi} ksin(nx)\,dx]$ $=\frac{1}{\pi}\left[-\frac{kcos(nx)}{n}|_{-\pi}^{0} + \frac{kcos(nx)}{n}|_0^{\pi}\right]$

We know $\cos(-\alpha) = \cos(\alpha)$ and $\cos(0) = 1$

$$b_n = \frac{k}{n\pi}[cos0 - \cos(-n\pi) - \cos(n\pi) - cos0] = \frac{2k}{n\pi}[1 - \cos(n\pi)]$$

Now $\cos(\pi) = -1,\ \cos(2\pi) = 1, \cos(3\pi) = -1,$

$\cos(n\pi) = \begin{cases} -1 & for\ odd\ n \\ +1 & for\ even\ n \end{cases}$ and

$1 - \cos(n\pi) = \begin{cases} 2 & for\ odd\ n \\ 0 & for\ even\ n \end{cases}$

The Fourier coefficients b_n of our function are:

$b_1 = \frac{4k}{\pi},\ b_2 = 0,\ b_3 = \frac{4k}{3\pi},\ \ b_4 = 0,\ b_5 = \frac{4k}{5\pi}, \dots \dots$

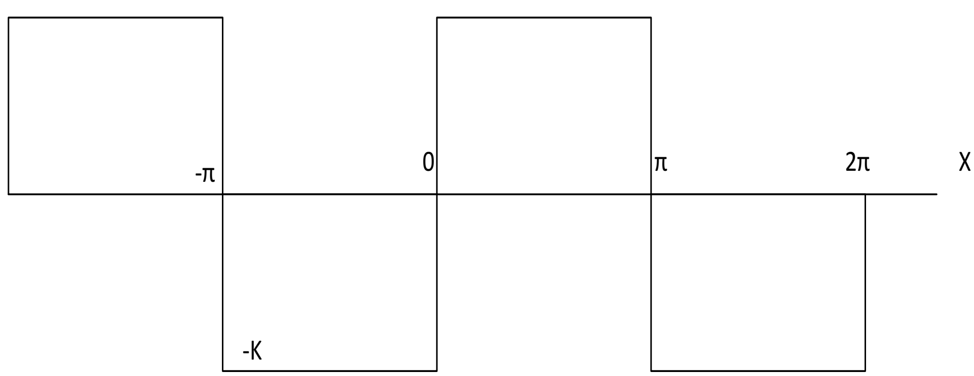

Figure 4

Since a_n are 0, the Fourier series of $f(x)$ is

$\frac{4k}{\pi}(sinx + \frac{1}{3}sin3x + \frac{1}{5}sin5x + \ldots..)$

Ex. 3: Find the Fourier coefficients of the periodic function $f(x)$

$$f(x) = x^2 \qquad 0 < x < 2, \qquad 2T = 2, \qquad T = 1$$

$$a_0 = \frac{1}{2T}\int_0^{2T} f(x)dx = \frac{1}{2.1}\int_0^{2T} x^2 dx = \frac{x^3}{2.3}\Big|_0^2 = \frac{8}{2.3}$$

$$a_n = \frac{1}{T}\int_0^{2T} f(x)\cos\left(\frac{n\pi}{T}x\right)dx = 1\int_0^2 x^2\cos\left(\frac{n\pi}{1}x\right)dx$$

$$a_n = \left(x^2\frac{\sin(n\pi x)}{n\pi} + 2x\frac{-\cos(n\pi x)}{n^2\pi^2} + 2\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\Big|_0^2$$

$$a_n = \left(x^2\frac{\sin(0)}{n\pi} + 2x\frac{-\cos(0)}{n^2\pi^2} + 2\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\Big|_0^2 = 4\frac{\cos(2n\pi)}{n^2\pi^2} = \frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{T}\int_0^{2T} f(x)\sin\left(\frac{n\pi}{T}x\right)dx = 1\int_0^2 x^2\sin\left(\frac{n\pi}{1}x\right)dx$$

$$b_n = \left(x^2\frac{-\cos(n\pi x)}{n\pi} - 2x\frac{-\sin(n\pi x)}{n^2\pi^2} + 2\frac{\cos(n\pi x)}{n^3\pi^3}\right)\Big|_0^2$$

$$b_n = \left(\frac{-4\cos(2n\pi)}{n\pi} + 2\frac{\cos(n\pi x)}{n^3\pi^3} - \frac{-2}{n^3\pi^3}\right) = \frac{-4}{n\pi}$$

$$f(x) = \frac{8/3}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\cos n\pi x - \frac{-4}{n\pi}\sin n\pi x\right)$$

1.1 Theorem: The Computation of the usual Fourier series is based on the integral identities:

a) $\int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,dx = 0 \ \ (n \neq m)$

b) $\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx = 0 \ \ (n \neq m)$

c) $\int_{-\pi}^{\pi}\sin(nx)\cos(mx)\,dx = 0 \ \ (n \neq m \text{ or } n = m)$

Proof: $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x\,dx =$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x\,dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x\,dx$$

Since $m \neq n$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)x\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin(n-m)x\,dx = 0 + 0$$

Using equations (1) (2) (3), we have

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)]dx = a_0.2\pi$$

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}(a_n \int_{-\pi}^{\pi} \cos nx\,dx + b_n \int_{-\pi}^{\pi} \sin nx\,dx)$$

Then:

$$\int_{-\pi}^{\pi} f(x)cosmxdx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)]cosmxdx = a_m \int_{-\pi}^{\pi} cos^2(mx)dx$$

$$\int_{-\pi}^{\pi} f(x)sinmxdx = \int_{-\pi}^{\pi} [a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)]sinmxdx = b_m \int_{-\pi}^{\pi} sin^2(mx)dx$$

1.2 Functions of Period 2L

For a function $f(x)$ periodic on an interval $[-L, L]$ instead of $[-\pi, \pi]$ a simple change of variables can be used to$x = \frac{\pi \acute{x}}{L}$, $dx = \frac{\pi d\acute{x}}{L}$

Solving for $\acute{x}$ gives $\acute{x} = \frac{Lx}{\pi}$ and plugging this in gives:

$$f(\acute{x}) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi\acute{x}}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi\acute{x}}{L}$$

Therefore; $a_0 = \frac{1}{2L}\int_{-L}^{L} f(\acute{x})d\acute{x}$ $a_n = \frac{1}{L}\int_{-L}^{L} f(\acute{x})\cos(\frac{n\pi\acute{x}}{L})d\acute{x}$ $b_n = \frac{1}{L}\int_{-L}^{L} f(\acute{x})\sin(\frac{n\pi\acute{x}}{L})d\acute{x}$

_Ex:_ The function is instead defined on the interval $[0,2L]$ then:

$$a_0 = \frac{1}{2L}\int_{0}^{2L} f(\acute{x})d\acute{x}$$

$$a_n = \frac{1}{L}\int_{0}^{2L} f(\acute{x})\cos(\frac{n\pi\acute{x}}{L})d\acute{x}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(\acute{x})\sin(\frac{n\pi\acute{x}}{L})d\acute{x}$$

1.3 Even and Odd functions, Half Range Expansions:

If a function is even so that $f(x) = f(-x)$ then $f(x)\sin(nx)$ is odd. Therefore, $b_n = 0$ for all n

Similarly, if a function is odd so that $f(x) = -f(-x)$ then $f(x)\cos(nx)$ is odd. Therefore,

$a_n = 0$ for all n

Even function	Odd function
Cosine harmonic	Sine harmonic
$b_n = 0$	$a_0 = 0,$ $a_n = 0$
$a_0 = \frac{1}{\pi}\int_0^{\pi} F(x)dx$	$b_n = \frac{2}{\pi}\int_0^{\pi} F(x)\sin(nx)dx$
$a_n = \frac{2}{\pi}\int_0^{\pi} F(x)\cos(nx)dx$	$F(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

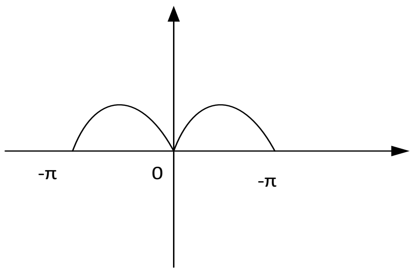

Figure 5

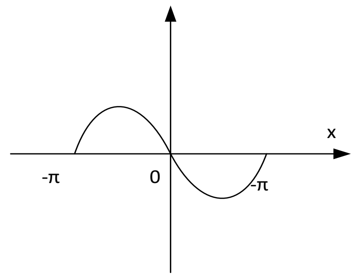

Figure 6

Notes:

$sinn\pi = 0$	$\cos(n\pi) = (-1)^n$
$if\ n = 1 \to sin180 = 0$	$cos180 = -1$
$if\ n = 2 \to sin360 = 0$	$cos360 = 1$
$sin2n\pi = 0$	$cos2n\pi = 1$
$\sin\ ((2n \pm 1)\pi) = 0$	$\sin\big((2n \pm 1)\pi\big) = -1$
$\sin\ (\frac{(2n-1)\pi}{2}) = (-1)^{n+1}$	$\cos\left(\frac{(2n-1)\pi}{2}\right) = 0$

Ex: Since $v = \frac{\pi}{L}x$ we have $\frac{\pi}{L}dx$

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L})$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \ , \ \ n = 1,2, \ldots.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \ , \ \ n = 1,2, \ldots.$$

Ex: Find the Fourier series of the function

$$f(x) = \begin{cases} 0 \ \ if \ \ -2 < x < -1 \\ k \ \ if -1 < x < 1 \\ 0 \ if \ 1 < x < 2 \end{cases} \qquad \begin{array}{c} p = 2L = 4 \\ L = 2 \end{array}$$

Solution: $a_0 = \frac{k}{2}$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \ = \frac{1}{2}\int_{-1}^{1} \mathrm{k}\cos\left(\frac{n\pi x}{2}\right)dx = \ \frac{k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Thus $a_n = 0 \ if \ n \ is \ even$ and

$$a_n = \frac{2k}{n\pi} \ if \ n = 1,5,9, \ldots..$$

$$a_n = \frac{-2k}{n\pi} \ if \ n = 3,7,11, \ldots..$$

We have $b_n = 0, \ \ n = 1,2, \ldots.$

$$f(x) = \ \frac{k}{2} + \ \frac{2k}{\pi}[\cos\left(\frac{\pi}{2}x\right) - \frac{1}{3}\cos\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\cos\left(\frac{5\pi}{2}x\right) - \cdots + \cdots]$$

Ex: Find the Fourier series function

$$f(x) = \begin{cases} -k & if \ \ -2 < x < 0 \\ \ \ k & if \ \ \ 0 < x < 2 \end{cases} \qquad p = 2L = 4, \quad L = 2$$

Solution: Since $L = 2$ we have $v = \frac{\pi x}{2}$

$$g(v) = \frac{4k}{\pi}[sinv + \frac{1}{3}sin3v + \frac{1}{5}\text{sin5v} + \cdots]$$

$$f(x) = \frac{4k}{\pi}[sin\frac{\pi x}{2} + \frac{1}{3}sin\frac{3\pi x}{2} + \frac{1}{5}\text{sin5}\frac{\pi x}{2} + \cdots]$$

1.4 Harmonic Analysis:

$$f(x) = \frac{a_0}{2} + a_1 cosx + b_1 sinx + a_2 cos2x + b_2 sin2x + \cdots$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = 2.\frac{1}{2.2\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$f(x) = a_0 + a_1 cosx + b_1 sinx$$

$$f(x) = a_0 + a_1 cosx + b_1 sinx + a_2 cos2x + b_2 sin2x$$

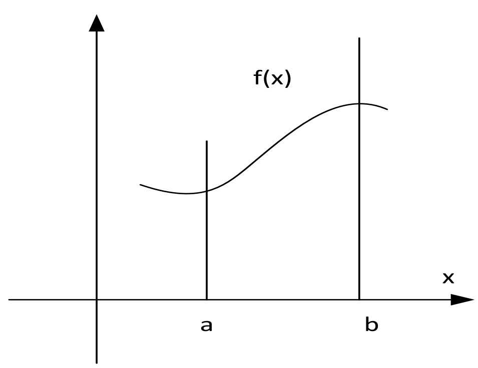

Figure 7

<u>Simplifications: even and odd functions</u>

If $f(x)$is an even function that is $f(-x) = f(x)$ then it is Fourier cosine series

$$\boldsymbol{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})} \qquad \textbf{if even}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

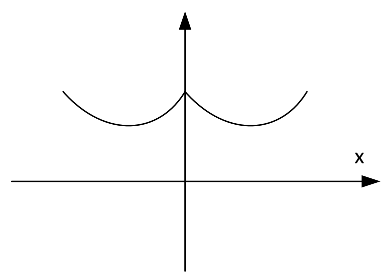

Figure 8

If $f(x)$is an odd function that is $f(-x) = -f(x)$ then it is Fourier sine series

$$\boldsymbol{f(x) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})} \qquad \textbf{if odd}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad n = 1,2,3,\ldots$$

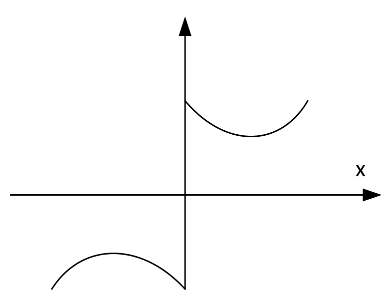

Figure 9

We have

$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$ for even g

$\int_{-L}^{L} h(x)dx = 0$ for odd h

Then

- Even function of period of 2π

  If f is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(x)dx$ $\quad$ $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)cosnx\, dx$ $\quad$ $n = 1,2,3, \ldots$

- Even function of period of 2π

  If f is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)sinnx\, dx$ $\quad$ $n = 1,2,3, \ldots$

<u>Ex. Find the Fourier series of the function</u>

$f(x) = x + \pi \quad if -\pi < x < \pi$ and

$f(x + 2\pi) = f(x)$

Solution: $f = f_1 + f_2 \quad f_1 = x\,,\ \ f_2 = \pi$

f_1 is odd $a_n = 0\ \ for\ n = 1,2,3, \ldots$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1(x) sinnx\, dx \quad = \frac{2}{\pi}\int_0^{\pi} x\ sinnx\, dx \quad = \frac{2}{\pi}\left[-\frac{xcosnx}{n}|_0^{\pi} + \frac{1}{n}\int_0^{\pi} cosnx\, dx \quad \right]$$

$$b_n = \frac{2}{n} cosn\pi$$

Hence $b_1 = 2,\ \ b_2 = -\frac{2}{2},\quad b_3 = \frac{2}{3},\ b_4 = -\frac{2}{4}, \ldots$

The Fourier series of $f(x)$ is

$$f(x) = \pi + 2(sinx - \frac{1}{2} sin2x + \frac{1}{3} sin3x - \cdots + \cdots)$$

<u>Half- Range Expansions</u>

<u>Ex. Find the two half-range expansions of the function</u>

$$f(x) = \begin{cases} \frac{2k}{L} x \quad if \qquad 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L - x) \quad if\, \frac{L}{2} < x < L \end{cases}$$

Solution:

- even periodic extension

$$a_0 = \frac{1}{L}[\frac{2k}{L}\int_0^{\frac{l}{2}} xdx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L - x)dx] = \frac{k}{2}$$

$$a_n = \frac{2}{L}[\frac{2k}{L}\int_0^{\frac{l}{2}} xcos(\frac{n\pi}{L}x)dx + \frac{2k}{L}\int_{\frac{L}{2}}^{L}(L - x)cos(\frac{n\pi}{L}x)dx] \quad \text{then}$$

$$\int_0^{\frac{l}{2}} xcos\left(\frac{n\pi}{L}x\right) dx = \left[\frac{Lxsin\left(\frac{nx}{L}x\right)}{n\pi}|_0^{\frac{L}{2}} - \frac{L}{n\pi}\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi}{L}x\right) dx \quad \right] =$$

$$\frac{L^2}{2\text{n}\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{\pi^2 n^2}(\cos\frac{n\pi}{2} \quad - 1)$$

$$\int_{\frac{L}{2}}^{L}(L-x)cos(\frac{n\pi}{L}x)dx] = \left[\frac{L(L-x)sin\left(\frac{nx}{L}x\right)}{n\pi}|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\left(\frac{n\pi}{L}x\right)dx\right] =$$

$$(0-\frac{L}{2\mathrm{n}\pi}\left(L-\frac{L}{2}\right)\sin\frac{n\pi}{2})-\frac{L^2}{\pi^2 n^2}(\cos n\pi \quad -\cos\frac{n\pi}{2})$$

$$a_n = \frac{4k}{\pi^2 n^2}(2\cos\frac{n\pi}{2} - \cos n\pi - 1)$$

$$a_2 = -\frac{16k}{12^2\pi^2}, \quad a_6 = -\frac{16k}{16^2\pi^2}, \quad a_{10} = -\frac{16k}{110^2\pi^2}, \dots\dots$$

And $a_n = 0 \quad if \ n \neq 2, 6, 10, \ 14 \dots\dots$

$$f(x) = \frac{k}{2} + \frac{16k}{\pi^2}(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots)$$

- Odd periodic extension

$$b_n = \frac{8k}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{8k}{\pi^2}(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x \dots)$$

1.5 Orthogonal Series: Generalized Fourier series

Let $y_0, \ y_1, \ y_2, \dots$. Be the orthogonal with respect to a weight function $r(x)$ on an interval $a \leq x$

and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots.$$

This is called an orthogonal series or generalized Fourier series.

Assume that integration is permissible

$(f, y_n) = \int_a^b r\, f\, y_n\, dx = \int_a^b r\, (\sum_{m=0}^{\infty} a_m y_m) y_n\, dx = \sum_{m=0}^{\infty} a_m \int_a^b r\, y_m y_n dx$=$\sum_{m=0}^{\infty} a_m (y_m, y_n)$

$a_n(y_n, y_n) = a_n \|y\|^2$ thus $(f, y_n) = a_n \|y\|^2$

$$a_m = \frac{(f, y_m)}{\|y_m\|^2} = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x)\, y_m(x) dx \quad m = 0,1,2, \ldots. \qquad (\#)$$

Example1: Fourier Legendre Series:

A Fourier Legendre series is an Eigen function expansion

$f(x) = \sum a_m p_m(x) = a_0 p_0 + a_1 p_1(x) + \cdots = a_0 + a_1 x + a_1 (\frac{3}{2} x^2 - \frac{1}{2}) + \cdots$

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x)\, p_m(x) dx \qquad m = 0,1,2, \ldots.$$

$$\|p_m\| = \sqrt{\int_{-1}^{1} p_m\, (x)^2 dx} = \sqrt{\frac{2}{2m+1}} \qquad m = 0,1,2, \ldots.$$

Ex: if $a_m = \frac{2m+1}{2} \int_{-1}^{1} \sin \pi x\;\; p_m(x) dx$ thus:

$a_m = \frac{3}{2} \int_{-1}^{1} \mathrm{x} \sin \pi x\;\; dx = \frac{3}{\pi} = 0.95493$

Then the Fourier Legendre series of $sin\pi x$ is

$sin\pi x = 0.95493 p_1(x) - 1.15824 p_3(x) + 021929 p_5(x) - 0.01664 p_7(x) + \cdots$

Example2: Fourier Bessel series

For each fixed nonnegative integer n, the sequence of Bessel functions of the first kind

$J_n(K_{n,1} x), J_n(K_{n,2}\, x)$, … with $K_{n,m}$, $kk = \alpha_{n,m}$ $thus\ k_{n,m} = \frac{\alpha_{n,m}}{R}$ $\qquad m = 1,2, \ldots.$

On the interval $0 \leq x \leq R$ with respect to the weight function $r(x) = x$ that is

$$\int_0^R x\, J_n\big(K_{n,m}x\big) J_n\big(K_{n,j}x\big)\, dx = 0 \qquad (j \neq m, n\ fixed)$$

The orthogonal set for J_n is $J_n\big(K_{n,1}x\big), J_n\big(K_{n,2}\,x\big),\ J_n\big(K_{n,3}\,x\big), \ldots$ n is fixed and $K_{n,m}$ is given

$$KR = \alpha_{n,m}\ , K_{n,m} = \frac{\alpha_{n,m}}{R}$$

The Fourier Bessel series to J_n is

$$f(x) = \sum a_m\ J_n(K_{n,m}x) = \ a_1\ J_n(K_{n,1}x) +\ a_2\ J_n(K_{n,2}x)\ \ + \cdots$$

$$a_m = \frac{2}{R^2 {J_{n+1}}^2 \alpha_{n,m}} \int_0^R x f(x)\ J_n(K_{n,m}x) dx \qquad m = 0,1,2, \ldots.$$

$$\left\| J_n(K_{n,m}x) \right\|^2 = \int_0^R x\, {J_n}^2 (K_{n,m}x) dx = \frac{R^2}{2}\, {J_{n+1}}^2 (K_{n,m}R)$$

<u>Example3: Fourier Bessel series</u>

Let $f(x) = 1 -\ x^2 \qquad R = 1, \quad n = 0$

$$a_m = \frac{2}{{J_1}^2(\lambda)} \int_0^1 x(1 - x^2)\, J_0(\lambda x) dx = \ \frac{2}{{J_1}^2(\lambda)} \left[\frac{1}{\lambda}(1 - x^2) x\, J_1(\lambda x)|_0^1 - \frac{1}{\lambda} \int_0^1 x(-2x)\, J_1(\lambda x) dx \right]$$

$$a_m = \frac{4 J_2(\lambda)}{\lambda^2 {J_1}^2(\lambda)} \qquad (\lambda = \alpha_{0,m})$$

$$J_2 = 2x^{-1} J_1 - J_0$$

1.6 Double Fourier Series:

If we have a function of two variables, then we can extend the general Fourier series as the double Fourier in the double Fourier series we can discuss two variables simultaneously. Let $f(x,y)$a function where $x \; and \; y$ are defined in the intervals $(-\pi,\pi) \; i.e. -\pi < x < \pi \; and$ $-\pi < y < \pi$ so we can define the general term of the double Fourier series of the function $f(x,y)$ as

$$A_{i,j} = \varepsilon_{i,j}\{a_{i,j}cosix \; cosjy + b_{i,j}sinix \; sinjy + \; c_{i,j}cosix \; sinjy + \; d_{i,j}sinix \; cosjy\}$$

$$\varepsilon_{0,0} = \frac{1}{4}, \quad \varepsilon_{i,0} = \frac{1}{2} \quad when \; i > 0$$

$$\varepsilon_{o,j} = \frac{1}{2} \quad when \; j > 0 \quad and \; \varepsilon_{i,j} = 1 \quad when \; i,j > 0$$

And we can find the coefficients of the Double Fourier series by term integration as same as the general Fourier series with the help of below formula

$$a_{i,j} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)cosix \; cosjy \; dx \; dy$$

<u>*Ex: Support $f(x,y), g(x,y)$ are two real valued functions.*</u> Since they are dependent on two variables $x, amd \; y$ they shall be defined on the rectangle

$$R: \; 0 \le x \le L_1 \quad and \quad 0 \le y \le L_2$$

$$(f|g) = \int_0^{L_1}\int_0^{L_2} f(x,y)g(x,y) \; dx \; dy$$

The functions $f_{m,n} \; and \; g_{i,j}$ are orthogonal if

$$(f_{m,n} |\ g_{i,j}) = \int_0^{L_1} \int_0^{L_2} f_{m,n}(x,y)\ g_{i,j}(x,y) dx\ dy = \begin{cases} 0 & for\ m \neq n\ \ or\ i \neq j \\ 1 & for\ m =\ n\ \ or\ i \neq j \end{cases}$$

Complex Fourier series:

Let $f(x)$ is the given function which is defined on the interval $(-\pi, \pi)$ we know that the cos and sin functions in terms of exponential

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad\qquad sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$

So we can write the Fourier series for the given complex function as:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \frac{e^{i\theta} + e^{-i\theta}}{2} + b_k \frac{e^{i\theta} + e^{-i\theta}}{2i} \right)$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{a_k - ib_k}{2}\ e^{ikx} + \sum_{k=1}^{\infty} \frac{a_k + ib_k}{2}\ e^{-ikx} = \sum_{k=-\infty}^{\infty} ck\ e^{ikx}$$

We assume that notations as follows:

$$c_0 = \frac{a_0}{2}\ ,\ \ c_k = \frac{a_k - ib_k}{2} \quad and \quad c_{-k} = \frac{a_k + ib_k}{2}$$

Here coefficients c_k are called the complex Fourier coefficients and they are defined

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx}\ dx \qquad where\ k = 0, \pm 1, \pm 2, \dots \dots$$

2-Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L \quad , \quad t > 0$$

To find u, we will solve this equation subject to the boundary conditions

- $u(t,0) = 0 \;\; and \; u(L,t) = 0 \quad for\; all\; t > 0$

$$u_t(0,t) = 0 \;\; and \; u_t(L,t) = 0 \quad for\; all\; t > 0$$

and the initial conditions

- $u(x,0) = f(x) \;\; and \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad for \quad 0 < x < L$

Separating variables:

<u>Step1:</u> $u(x,t) = X(x)\,T(t)$

$$\frac{\partial^2 u}{\partial t} = XT'' \quad , \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

$$XT'' = c^2 X''T \quad then \quad \frac{T''}{c^2 T} = \frac{X''}{X}$$

$$\frac{T''}{c^2 T} = K \quad , \frac{X''}{X} = K$$

K is an arbitrary constant called the separation constant

$$X'' - KX = 0, \qquad T'' - Kc^2 T = 0$$

Now we get:

$$X(0)T(t) = 0 \quad and \; X(L)T(t) = 0 \quad for\; all\; t > 0$$

$if\ X(0) \neq 0\ or\ X(L) \neq 0\ then\ \ T(t)$ must be 0 for all t and u is identically zero.

Then $X(0) = 0 \ \ and\ X(L) = 0$

$X'' - KX = 0\,, \qquad X(0) = 0\ , \qquad X(L) = 0$

Step2: Solving the separable equations if K is positive say $K = \mu^2$ with $\mu > 0$ then

$X'' - \mu^2 X = 0$

$X(x) = c_1 \cosh\ \mu x\ +\ c_2 sinh\ \mu x \qquad , X(0) = 0$

$0 = c_1 \cosh\ 0\ +\ c_2 sinh\ 0\ = c_1$

$X(x) =\ c_2 sinh\ \mu x \qquad\qquad , X(L) = 0$

$c_2\ sinh\ \mu x\ \ = 0\ \ but\ \ \ \mu L \neq 0$ so $\sinh \mu L \neq 0$ and hence $c_2 = 0\ implying$ that $X = 0$ this case $K > 0$

If $K = 0, X'' = 0$ general solution

$X(x) =\ c_1\ x + c_2$

We have to take $c_1\ = c_2\ = 0$

$u = 0 \qquad K = -\mu^2 < 0$

The corresponding boundary value problem in X is

$X'' - \mu^2 X = 0 \qquad\qquad , X(0) = 0,\ \ X(L) = 0$

The general solution of the differential equation is $X = c_1\ \cos \mu x +\ c_2\ \sin \mu x$

The condition $X(0) = 0,\ \ c_1\ = 0\,, hence\ X = \sin \mu x$

The condition $X(L) = 0,\ c_2\ \sin \mu x = 0$

The trivial solution $X = 0\ we\ take\ c_2\ = 1\ then\ \sin \mu L = 0$

$\mu = \mu_n = \frac{n\pi}{L}, \quad n = \pm 1, \pm 2, \dots..$

$X = X_n = sin\frac{n\pi}{L}x\ , \qquad n = 1,2, \dots.$

We now have $K = -\mu^2 = -(\frac{n\pi}{L})^2$ and get

$$\boldsymbol{T'' + (c\frac{n\pi}{L})^2 T = 0}$$

The general solution of this equation is

$$T_n = b_n cos\lambda_n t + a_n sin\lambda_n t$$

$$\boldsymbol{\lambda_n = c\frac{n\pi}{L}\ , \quad n = 1, 2, \dots}$$

$$\boldsymbol{u_n(x,t) = sin\frac{n\pi}{L}x\,(b_n cos\lambda_n t + a_n sin\lambda_n t) \qquad n = 1, 2 \dots}$$

$$\boldsymbol{u(x,t) = \sum_{n=1}^{\infty} sin\frac{n\pi}{L}x\,(b_n cos\lambda_n t + a_n sin\lambda_n t)}$$

Step 3: Fourier series solution of the Entire problem

$u(x,0) = f(x) = \sum_{n=1}^{\infty} \left(b_n\ sin\frac{n\pi}{L}x\right) \qquad , 0 < x < L$

The series on the right is the half-range sine series expansion of f

$$b_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi}{L}x \ dx \quad n = 1,2,\ldots$$

Similarly, for a_n

$$g(x) = \sum_{n=1}^{\infty}\left(a_n\lambda_n \ sin\frac{n\pi}{L}x\right)$$

$$a_n^*\lambda_n = \frac{2}{L}\int_0^L g(x) sin\frac{n\pi}{L}x \ dx \quad n = 1,2,\ldots$$

Now $a_n^* = \frac{2}{cn\pi}\int_0^L g(x) sin\frac{n\pi}{L}x \ dx \quad n = 1,2,\ldots$

2.1 Solution of the one Dimensional Wave equation:

The solution of the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L\,, \ \ t > 0$$

With boundary conditions

$$u(0,t) = 0 \quad and \ u(L,t) = 0 \quad for \ all \ t > 0$$

And initial conditions

$$u(x,0) = f(x), \ \ \frac{\partial u}{\partial t}(x,0) = g(x) \ \ for \ 0 < x < L$$

$$u(x,t) = \sum_{n=1}^{\infty} sin\frac{n\pi}{L}x\,(b_n cos\lambda_n t + \ a_n sin\lambda_n t)$$

Where $b_n = \frac{2}{L}\int_0^L f(x) sin\frac{n\pi}{L}x \ dx \quad , a_n = \frac{2}{cn\pi}\int_0^L g(x) sin\frac{n\pi}{L}x \ dx, \ \ \lambda_n = c\frac{n\pi}{L}\ , \quad n = 1,2,\ldots$

2.2 Heat Equation:

The one Dimensional Heat Equation. Let $u(x,t)(0 < x < L, t > 0)$ represent the temperature of the point x of the bar at time t. Given that the initial temperature distribution of the bar is $u(x,0) = f(x)$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L\,,\ \ t > 0$$

In addition, u satisfies the boundary conditions $u(t,0) = 0\,, u(L,t) = 0 \ \ for\ all\ t > 0$

And the initial condition $u(x,0) = f(x) \quad for\ 0 < x < L$

- Separation of variables

$u(x,t) = X(x)T(t)$ where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone

$\frac{T'}{c^2T} = \frac{X''}{X}$

$$\frac{T'}{c^2T} = K \quad , \frac{X''}{X} = K$$

$X'' - KX = 0 \quad , T' - \ Kc^2T = 0$ we get $X(0)T(t) = 0,\ \ X(L)T(t) = 0\ forall\ t > 0$

$X(0)T(t) = 0\ , \quad X(L) = 0$ we thus

$X'' - KX = 0\ , \quad X(0) = 0\,,\ \ X(L) = 0$ we found that

$$K = -\mu^2 \qquad \mu = \mu_n = \frac{n\pi}{L}, \quad n = 1,2,\ldots$$

$$X = X_n = sin\frac{n\pi}{L}x\,, n = 1,2,\ldots$$

$$T' - \ (c\frac{n\pi}{L})^2T = 0$$

$T_n(t) = b_n e^{\lambda n^2 t} \qquad n = 1,2, ...$

$\lambda_n = \frac{n\pi}{L} \qquad n = 1,2, ...$

$u_n(x,t) = b_n e^{-\lambda n^2 t \, sin \frac{n\pi}{L} x} \qquad n = 1,2, ...$

$$\boldsymbol{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda n^2 t \, sin \frac{n\pi}{L} x}}$$

2.3 Introduction to separation of variables and heat equation

2.3.1 Separation of variables

This method can be used to solve many kinds of linear PDE, but as a typical example we will consider the heat equation

$4u_{xx} = u_t \qquad 0 < x < \pi, \qquad t > 0 \qquad (4)$

$u(0,t) = 0 \qquad t > 0 \qquad (5)$

$u(\pi,t) = 0 \qquad t > 0 \qquad (6)$

$u(x,t) = f(x) \qquad 0 \le x \le \pi \qquad (7)$

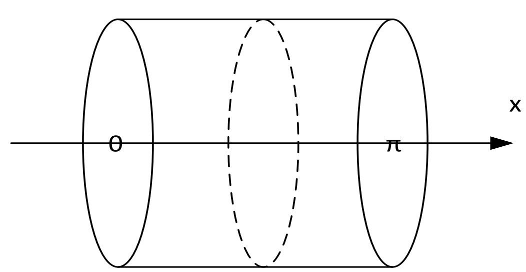

Figure 10

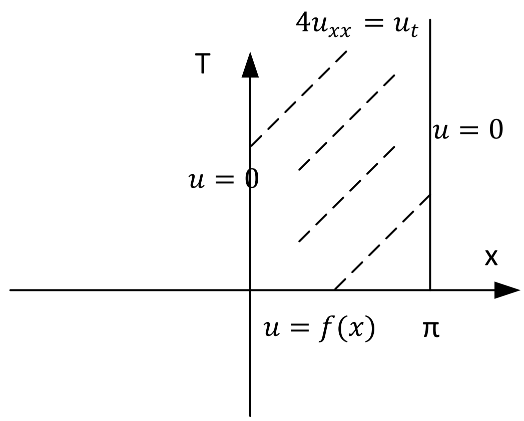

Figure 11

The idea is to assume the PDE (1) has a special form

$u(x,t) = X(x)T(t)$ (7)

Form (#) we have $u_x = X'(x)T(t)$

u_{xx}=$X''(x)T(t)$ and u_t=$X(x)T'(t)$ so our PDE (1) becomes

$4u_{xx}$= $u_t \Rightarrow\ 4X''(x)T(t) = X(t)T'(t)$

separate the variables to form

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{4T(t)} \qquad (8)$$

Now since LHS of (8) only depends on x and since RHS of (8) only depends on t, we must have a constant γ such that

$$\boldsymbol{\frac{X''}{X} = \frac{T'}{4T} = \gamma}$$

We form two ODEs

$T' = 4\,\gamma T \quad --\rightarrow\ T(t) = ce^{4\,\gamma T} \quad , c\ constant$

$X'' - \gamma X = 0 --\rightarrow char\ equ \quad \lambda^2 = \gamma$ thus

$$X(x) = \begin{cases} Ax + B\,, & \gamma = 0 \\ Ae^{\sqrt{\gamma}x} + Be^{-\sqrt{\gamma}x}\,, & \gamma > 0\ \{roots\ are\ real\ \pm\sqrt{\gamma}\} \\ Acos\sqrt{-\gamma}x + Bsin\sqrt{-\gamma}x\,, & \gamma < 0\ \{roots\ are\ complex\ \pm i\sqrt{\gamma}\} \end{cases}$$

$$u(0,t) = X(0)T(t) = 0 \quad --\rightarrow \quad X(0) = 0 \qquad (9.1)$$

$$u(\pi,t) = X(\pi)T(\pi) = 0 \quad --\rightarrow \quad X(\pi) = 0 \qquad (9.2)$$

$T \equiv 0$ since this produces the trivial solution.

1- Case $\gamma = 0$ $\quad X(x) = Ax + B\ \ and\ (9.1\ \&9.2)$ yield

$X(0) = A.0 + B = 0 \quad so\ B = 0$

$X(\pi) = A.\pi + 0 = 0 \quad so\ A = 0$

But this yields the trivial solutions.

2- Case $\gamma > 0$ $\quad X(x) = \ Ae^{\sqrt{\gamma}x} + Be^{-\sqrt{\gamma}x}\ \ and\ (9.1\ \&9.2)$ yield

$X(0) = A + B = 0 \ \ so\ A = -B$

$X(\pi) = Ae^{\sqrt{\gamma}\pi} + Be^{-\sqrt{\gamma}\pi} = 0$ we form

$A(e^{\sqrt{\gamma}\pi} - \ e^{-\sqrt{\gamma}\pi}) = 0 \quad so\ A = 0$

But yields the trivial solution.

3- Case $\gamma < 0$ $\quad X(x) = Acos\sqrt{-\gamma}x + Bsin\sqrt{-\gamma}x$ and (9.1 &9.2) yield

$X(0) = A = 0 \ \ so\ A = 0$

$X(\pi) = Bsin\sqrt{-\gamma}\pi = 0$

But $B = 0$ (trivial solution) and consider $Bsin\sqrt{-\gamma}\pi = 0$ which holds when

$$\sqrt{-\gamma}\pi = n\pi \quad n = 1,2, \ldots$$

So we have

$$\boldsymbol{\sqrt{-\gamma} = n \quad n = 1, 2, \ldots}$$

$X_a(x) = \ B_a \sin nx \quad n = 1,2, \ldots. \qquad B_n \ constant$

$T(t) = ce^{4\gamma t}$ to form

$T_n(t) = c_n e^{-4a^2 t} \qquad n = 1,2, \ldots. \qquad c_n \ constant$

From our assumed form $u(x,t) = X(x)T(t)$, we form a sequence

$u_n(x,t) = X_n(x)\, T_n(t) \qquad n = 1,2, \ldots..$

$u_n(x,t) = (B_a \sin nx)\left(c_n e^{-4n^2 t}\right) = \ b_n \ e^{-4n^2 t} \sin nx \qquad (b_n = B_n c_n)$

Since (4), (5) and (6) are linear +homog every sum of solution is also a solv, hence we can form

$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \ e^{-4n^2 t} \sin nx \qquad$ (10)

To define b_n we use

$\boldsymbol{u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin nx} \qquad$ **(#)**

2.4 How we use Fourier series:

We can write our initial temperature $u(x,0) = f(x)$ as an infinite sum involving $\sin nx$ then we may calculate the b_n and hence determine the exact solution to the problem.

If $u(x,0) = f(x) = \sin x - 2 \sin 3x$ then we may write it as

$$u(x,0) = f(x) = \sin x - 2\sin 3x = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots$$

And equating the coefficients we see that $b_1 = 1, b_3 = -2$ and all the other b_i must be zero, thus our solution $\sum_{n=1}^{\infty} b_n \ e^{-4n^2 t} \sin nx$

$$u(x,t) = \ e^{-4t} \sin x - 2 \ e^{-36t} \sin 3x$$

However, if we had an initial temperature function of the form $u(x,0) = f(x) = X$ then to determine the b_n we would consider

$$u(x,0) = f(x) = X = \sum_{n=1}^{\infty} b_n \sin nx$$

This question leads Fourier to develop a type of series known as Fourier series involving sin and cos.

The b_n can be calculated from $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

Bar with insulated ends

Ex: consider the boundary value problem

$$u_t = u_{xx} \qquad 0 < x < L \qquad t > 0 \qquad (11)$$

$$u_x(0,t) = 0 \qquad u_x(L,t) = 0 \qquad t > 0 \qquad (12)$$

$$u(X,0) = h(x0 = X(L-x) \qquad 0 \le x \le L \qquad (13)$$

This determines the temperature distribution $u(x,t)$ in a rod of length L insulated laterally and at both ends $X = 0$, $X = L$ and with the initial temperature $h(x)$. Assume solv u to (1) is of form $u(x,t) = X(t)T(t)$ we have

$$u_{xx}(x,t) = X''(x)T(t) \qquad , \qquad u_t(x,t) = X(x)T'(t)$$

Hence (11) becomes $X(x)T'(t) = X''(x)T(t)$ and we separate the variables to form

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = \gamma$$

Where γ is the "separation constant" we form two ODEs

$T' = \gamma T \ \rightarrow T(t) = ce^{\gamma t}$

$X'' - \gamma X = 0$, we form the cherecve $\boldsymbol{\lambda^2 = \gamma}$ Thus:

$$X(x) = \begin{cases} Ax + B\,, & \gamma = 0 \\ Ae^{\sqrt{\gamma}x} + Be^{-\sqrt{\gamma}x}\,, & \gamma > 0 \\ A\cos\sqrt{-\gamma}x + B\sin\sqrt{-\gamma}x\,, & \gamma < 0 \end{cases}$$

Now (12) and $u = X.T$ yields

$0 = u_x(0,t) = X'(0)T(t) \ \rightarrow \qquad X'(0) = 0$

$0 = u_x(L,t) = X'(L)T(t) \ \rightarrow \qquad X'(L) = 0$

1- Case $\gamma = 0, A = 0$ *and* B *arbitrary so* $X \equiv B$

2- Case $\gamma > 0, A\sqrt{\gamma} - B\sqrt{\gamma} = 0 \ \Rightarrow A = B$

$$A\sqrt{\gamma}\, e^{\sqrt{\gamma L}} - B\sqrt{\gamma}\, e^{-\sqrt{\gamma L}} = 0$$ so we form

$A\sqrt{\gamma}\left[e^{\sqrt{\gamma L}} - \frac{1}{e^{\sqrt{\gamma L}}}\right] \Rightarrow A = 0$ *so* $B = 0$ Trival case

3- Case $\gamma < 0, \qquad -\sqrt{-\gamma}\, A\sin 0 + \sqrt{-\gamma}\, B\cos 0 = 0$

$$-\sqrt{-\gamma}\, A\sin\sqrt{-\gamma}\, L + \sqrt{-\gamma}\, B\cos\sqrt{-\gamma}\, L = 0$$

And we have $\sin\sqrt{-\gamma}\ L = 0$ which has solution $\sqrt{-\gamma}L = n\pi \quad n = 1,2,\ldots$

And hence $\sqrt{-\gamma} = \frac{n\pi}{L} \quad n = 1,2,\ldots$

Thus we for a sequence of functions

$$\boldsymbol{X_n(x) = A_n \cos\frac{n\pi x}{L} \qquad n = 1, 2, \ldots.}$$

And

$$\boldsymbol{T_n(t) = c_n e^{-(\frac{n\pi}{L})^2 t} \qquad n = 1, 2, \ldots.}$$

Hence $u_n(x,t) = X_n(x)T_n(t) \qquad n = 1,2, \ldots.$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty} u_n(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})\, e^{-(\frac{n\pi}{L})^2 t}$$

Now (13) yields $u(x,0) = h(x) == a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi x}{2})$

Now this is just a Fourier cosine series for h this

$$a_0 = \frac{1}{L}\int_0^L h(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^L h(x) \cos\frac{n\pi x}{L}\, dx$$

$$a_0 = \frac{1}{L}\int_0^L x(L-x)\, dx = \frac{1}{L}\int_0^L (Lx - x^2)\, dx = \frac{1}{L}[\frac{L}{2}x^2 - \frac{1}{3}x^3]_0^L = \frac{L^2}{6}$$

Since out temperature decays exponentially in the summoned we have

$$u(x,t)_{t\to\infty} \;\to\; a_0 = \frac{L^2}{6}$$

2.5 The two Dimensional Wave and Heat Equations:

Two dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad 0 < x < a, \qquad 0 < y < b, \qquad t > 0$$

Where $u = u(x,y,t)$ at the point (x,y) at time t

We have $u = u(x,y,t) = 0$ on the boundary for all $t \geq 0$

$$u(0,y,t) = 0\ ,\ u(a,y,t) = 0\ \ for\ 0 \leq y \leq b \quad and\ t \geq 0$$

The initial conditions are

$$u(x,y,0) = f(x,y) \; and \; \frac{\partial u}{\partial t}(x,y,0) = g(x,y)$$

$$u(x,y,t) = X(x)Y(y)T(t)$$

$$XYT'' = c^2(X''YT + XY''T)$$

$$\frac{T''}{c^2T} = \frac{X''}{X} + \frac{Y''}{Y}$$

$$\frac{T''}{c^2T} = -K^2 \quad and \; \frac{X''}{X} + \frac{Y''}{Y} = -K^2, \quad K > 0$$

$$T'' + \; K^2\, c^2T = 0$$

$$\frac{X''}{X} = \frac{Y''}{Y} - K^2$$

$$\frac{X''}{X} = \; -\mu^2, \qquad \frac{Y''}{Y} - K^2 \; = \; -\mu^2 \qquad\qquad \mu > 0$$

$$X'' + \mu^2X = 0, \qquad\quad Y'' + V^2Y = 0, \qquad V^2 = K^2 - \mu^2$$

$$X'' + \mu^2X = 0 \;, \qquad\quad X(0) = 0\,, \quad X(a) = 0$$

$$Y'' + V^2Y = 0 \;, \qquad\quad Y(0) = 0\,, \quad Y(b) = 0$$

$$T'' + \; c^2K^2\, T = 0 \qquad\qquad K^2 = V^2 + \mu^2$$

2.6 Solution of the separated Equations:

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$$

$$Y(y) = d \cos Vx + d_2 \sin Vx$$

$$T(t) = \; e_1 \cos cKt + e_2 \sin cKt \qquad (K^2 = V^2 + \mu^2)$$

We get $c_1 = 0$ and $c_2 \sin \mu a = 0$

$d_1 = 0$ and $d \sin Va = 0$ thus

$$\mu = \mu_m = \frac{m\pi}{a} \quad and\ V = V_n = \frac{m\pi}{b} \qquad m, n = 1,2, \dots$$

So $X_m(x) = \sin \frac{m\pi}{a} x$, $Y_n(y) = \sin \frac{m\pi}{b} y$

If $m = 0\ or\ n = 0$ the solutions are identically zero.

For $m, n = 1,2, \dots$ we have

$$K = K_{mn} = \sqrt{\mu^2{}_{mn}} = \sqrt{\mu^2{}_m + V^2{}_n} = \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$$

$$T(t) = T_{mn}(t) = B_{mn} \cos \lambda_{mn} t + C_{mn} \sin \lambda_{mn} t \, , \qquad \lambda_{mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

$$u_{mn}(x, y, t) = \sin \frac{m\pi}{a} x \, \sin \frac{n\pi}{b} y \, (B_{mn} \cos \lambda_{mn} t + C_{mn} \sin \lambda_{mn} t)$$

The function $u_{mn}$is called the normal modes of the two dimensional wave equation.

3- Double Fourier series solution of the entire problem:

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (B_{mn} \cos \lambda_{mn} t + C_{mn} \sin \lambda_{mn} t) \sin \frac{m\pi}{a} x \, \sin \frac{n\pi}{b} y$$

From the initial condition $u(x, y, o) = f(x, y)$ we get

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{mn}\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y$$

$$=\int_0^b\int_0^a \sin\frac{m\pi}{a}x\ \sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\ \sin\frac{m'\pi}{a}x\sin\frac{n'\pi}{b}y\,dxdy=0$$

If $(m,n)\neq(m',n')$ also if $(m,n)=(m',n')$ then we get

$$\int_0^b\int_0^a \sin^2\frac{m\pi}{a}x\sin^2\frac{n\pi}{b}y\ dxdy=\frac{ab}{4}$$

$B_{mn}=\frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ dxdy$ this is called the double Fourier series of f, we

get $g(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{mn}\lambda_{mn}\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y$

$$C_{mn}=\frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ dxdy$$

3.1 The solution of the two Dimensional wave equation

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(B_{mn}\cos\lambda_{mn}t+\ C_{mn}\sin\lambda_{mn}t)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y$$

Where $\lambda_{mn}=c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$

3.2 Solution of the two Dimensional Heat equation for a Rectangle:

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\qquad 0<x<a,\qquad 0<y<b,\qquad t>0$$

With boundary conditions

$$u(0,y,t)=u(a,y,t)=0\ ,\quad for\ 0<y<\ b\quad ,\qquad t>0$$

$$u(x,0,t)=u(x,b,t)=0\ ,\quad for\ 0<x<\ a\quad ,\qquad t>0$$

And initial condition $u(x, y, 0)f(x, y) \quad 0 < x < a \ , 0 < y < b$

$$(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(A_{mn}\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}ye^{-\lambda^2 mnt}$$

Where $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x, y)\sin\frac{m\pi}{a}x\ \sin\frac{n\pi}{b}y\ dxdy \qquad m, n = 1,2, \ldots$$

3-3 Laplace's Equation in Rectangular coordinates:

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right) \qquad 0 < x < L, \qquad t > 0$$

$$u_{xx} = 0 \qquad , \quad u(x) = c_1 x + c_2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < a \ , 0 < y < b$$

Now $u(x, 0) = f_1(x), \quad u(x, b) = f_2(x) \quad 0 < x < a$

$$u(0, y) = g_1(y), \quad u(a, y) = g_2(x) \quad 0 < y < b$$

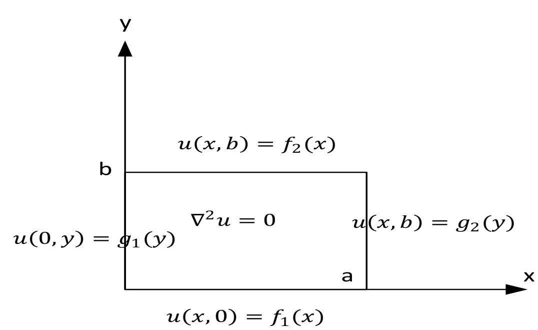

Figure 12

Ex: $X'' + KX = 0$, $Y'' + KY = 0$ K is the separation constant

$X(0) = 0$, $X(a) = 0$, $Y(0) = 0$

$K \leq 0$, $K = \mu^2 > 0$

$X = c_1 \cos \mu x + c_2 \sin \mu x$

$c_1 = 0$ $\mu = \mu_n = \frac{n\pi}{a}$ $n = 1,2, \ldots$

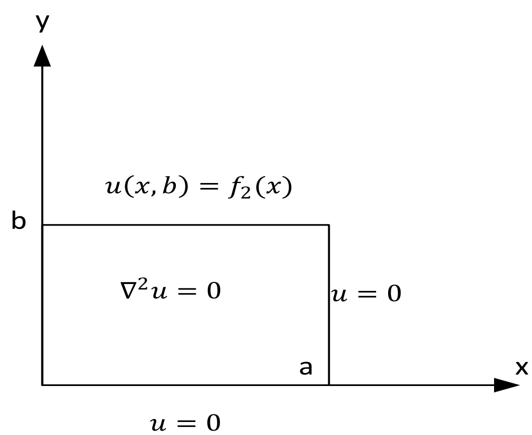

Figure 13

Hence $X_n(x) = \sin \frac{n\pi}{a} x$, $n = 1,2, \ldots$

Now to Y with $K = \mu^2{}_n$ we have

$Y = A_n \cosh \mu_n y + B_n \sinh \mu_n y$

$Y(0) = 0$ we find that $A_n = 0$

Hence $Y_n = B_n \sinh \mu_n y$

We get the general form of the solution:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$$

$$u(x,b) = f_2(x)$$

$f_2(x) = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi b}{a} \sin\frac{n\pi}{a}x$

$$B_n = \frac{2}{a\sinh\frac{n\pi b}{a}} \int_0^a f_2(x) \sin\frac{n\pi}{a}x\, dx \qquad n = 1,2, \ldots.$$

$$D_n = \frac{2}{b\sinh\frac{n\pi a}{b}} \int_0^b g_2(y) \sin\frac{n\pi}{b}y\, dy \qquad n = 1,2, \ldots.$$

$$A_n = \frac{2}{a\sinh\frac{n\pi b}{a}} \int_0^a f_1(x) \sin\frac{n\pi}{a}x\, dx \qquad n = 1,2, \ldots.$$

$$C_n = \frac{2}{b\sinh\frac{n\pi a}{b}} \int_0^b g_1(y) \sin\frac{n\pi}{b}y\, dy \qquad n = 1,2, \ldots.$$

Solution of the Dirichlet problem in a rectangle:

The solution of the two dimensional Dirichlet problem in figure is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}(b-y) + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{a}x \sinh\frac{n\pi}{a}y$$
$$+ \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi}{b}(a-x) \sin\frac{n\pi}{b}y + \sum_{n=1}^{\infty} D_n \sinh\frac{n\pi}{b}x \ \sin\frac{n\pi}{b}y$$

4-References

1- Differential Equations and Boundary Value Problems (Computing and Modeling), Fifth Edition, C. Henry Edward, David E. Penny, David T. Calvis, October 2014

2- Advanced Engineering Mathematics 10th edition, Erwin Kreyszig August 2011

3- Partial Differential Equations with Fourier Series and Boundary Value Problem, second Edition, Nakhle H. Asmar, 2004

4- Elementary Differential Equaions and Boundary Value Problems, 10th Edition, William E.. Boyce, Richard C. DiPrima, October 2012.